Nonextensive random matrix theory approach to mixed regular-chaotic dynamics

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We apply Tsallis' q-indexed entropy to formulate a nonextensive random matrix theory, which may be suitable for systems with mixed regular-chaotic dynamics. The joint distribution of the matrix elements is given by folding the corresponding quantity in the conventional random matrix theory by a distribution of the inverse matrix-element variance. It keeps the basis invariance of the standard theory but violates the independence of the matrix elements. We consider the subextensive regime of q more than unity in which the transition from the Wigner to the Poisson statistics is expected to start. We calculate the level density for different values of the entropic index. Our results are consistent with an analogous calculation by Tsallis and collaborators. We calculate the spacing distribution for mixed systems with and without time-reversal symmetry. Comparing the result of calculation to a numerical experiment shows that the proposed nonextensive model provides a satisfactory description for the initial stage of the transition from chaos towards the Poisson statistics.

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I. INTRODUCTION

The past decade has witnessed a considerable interest devoted to nonconventional statistical mechanics. Much work in this direction followed the line initiated by Tsallis' seminal paper [1]. The standard statistical mechanics is based on the Shannon entropy measure $S=-\Sigma_i p_i \ln p_i$ (we use Boltzmann's constant $k_B=1$), where $\{p_i\}$ denotes the probabilities of the microscopic configurations. This entropy is extensive. For a composite system $A+B$, constituted of two independent subsystems A and B such that the probability $p(A+B)=p(A)p(B)$, the entropy of the total $S(A+B)=S(A)+S(B)$. Tsallis proposed a nonextensive generalization: $S_q=(1-\Sigma_i p_i^q)/(q-1)$. The entropic index q characterizes the degree of extensivity of the system. The entropy of the composite system $A+B$, the Tsallis' measure verifies

$$S_q(A+B) = S_q(A) + S_q(B) + (1-q)S_q(A)S_q(B), \quad (1)$$

from which the denunciation nonextensive comes. Therefore $S_q(A+B) < S_q(A)+S_q(B)$ if $q>1$. This case is called subextensive. If $q<1$, the system is in the superextensive regime. The standard statistical mechanics recovered for $q=1$. Applications of the Tsallis formalism covered a wide class of phenomena; for a review please see, e.g., [2]. However, the relation between the parameter q and the underlying microscopic dynamics is not fully understood yet. The value of q has been obtained from studies of dynamics in cases of low-dimensional dissipative maps [3,4], and in some toy models of self-organized criticality [5]. Explicit expressions for q in terms of physical quantities exist in few cases, e.g., in turbulence problems [6] and physics of the solar plasma [7]. Aringazian and Mazhitov [8] obtained a Tsallis distribution function for a smaller subsystem weakly interacting with the remaining "quasi-thermostat" composed of a larger number M of particles, with an entropic index $q-1 \sim 1/M$.

A number of recent publications considered the possibility of a nonextensive generalization to the random matrix theory (RMT) [9]. This is the statistical theory of random matrices H whose entries fluctuate as independent Gaussian random numbers. The matrix-element distribution has been obtained by extremizing Shannon's entropy subject to the constraint of normalization and existence of the expectation value of $\mathrm{Tr}(H^\dagger H)$ [10]. What has become known as the Bohigas-Giannoni-Schmidt conjecture is that the quantum spectra of classically chaotic systems are correlated according to RMT, whereas the spectral correlations of classically regular systems are close to Poissonian statistics [11]. Several attempts have been made to extend the applicability of RMT to include quantum systems with mixed regular-chaotic classical dynamics; for a review please see [12]. For example, the principle of maximum entropy was used for this purpose by introducing additional constraints concerning the off-diagonal elements [13]. Nonextensive generalizations of RMT, on the other hand, extremize Tsallis' nonextensive entropy, rather than Shannon's. The first attempt in this direction is probably due to Evans and Michael [14]. Toscano *et al.* [15] constructed a non-Gaussian ensemble by minimizing Tsallis' entropy and obtained expressions for the level densities and spacing distributions. Bertuola *et al.* [16] have shown that Tsallis' statistics interpolate between RMT and an ensemble of Lévy matrices [17] that have a wide range of applications. They illustrated the spectral fluctuations in the subextensive regime by considering the gap function $E(s)$ that gives the probability of finding an eigenvalue-free segment of length s. Analytical expressions for the level-spacing distributions of mixed systems belonging to the three symmetry universality classes are obtained in [18]. A slightly different application of nonextensive statistical mechanics to RMT is due to Nobre and Souza [19].

In this work we use the integral representation of the gamma function to express the characteristics of the proposed nonextensive RMT in terms of integrals involving the characteristics of the conventional theory. We show that nonextensive statistics provides a principled way to accommodate systems with mixed regular-chaotic dynamics.

II. NONEXTENSIVE GENERALIZATION OF RMT

RMT replaces the Hamiltonian of the system by an ensemble of Hamiltonians whose matrix elements are independent random numbers. Dyson [20] showed that there are three generic ensembles of random matrices, defined in terms of the symmetry properties of the Hamiltonian. Time-reversal-invariant quantum system are represented by a Gaussian orthogonal ensemble (GOE) of random matrices when the system has rotational symmetry and by a Gaussian symplectic ensemble (GSE) otherwise. Chaotic systems without time reversal invariance are represented by the Gaussian unitary ensemble (GUE). The dimension β of the underlying parameter space is used to label these three ensembles: for GOE, GUE, and GSE, β takes the values 1, 2, and 4, respectively. Balian [10] derived the weight functions $P_\beta(H)$ for the three Gaussian ensembles from the maximum entropy principle postulating the existence of a second moment of the Hamiltonian. He applied the conventional Shannon definition for the entropy to ensembles of random matrices as $S=-\int dHP_\beta(H)\ln P_\beta(H)$ and maximized it under the constraints of normalization of $P_\beta(H)$ and fixed mean value of $\mathrm{Tr}(H^\dagger H)$. He obtained $P_\beta(H)\propto\exp[-\eta\,\mathrm{Tr}(H^\dagger H)]$ which is a Gaussian distribution with inverse variance $1/2\eta$. In this section, we apply the maximum entropy principle, with Tsallis' entropy, to random-matrix ensembles belonging to the three canonical symmetry universalities. The Tsallis entropy is defined for the joint matrix-element probability density $P_\beta(q,H)$ by

$$S_q[P_\beta(q,H)] = \left(1 - \int dH[P_\beta(q,H)]^q\right) \Big/ (q-1). \quad (2)$$

We shall refer to the corresponding ensembles as the Tsallis orthogonal ensemble (TsOE), the Tsallis unitary ensemble (TsUE), and the Tsallis symplectic ensemble (TsSE). For $q\to 1$, S_q tends to Shannon's entropy, which yields the canonical Gaussian orthogonal, unitary, or symplectic ensembles (GOE, GUE, GSE) [9,10].

There is more than one formulation of nonextensive statistics which mainly differ in the definition of the averaging. Some of them are discussed in [21]. We apply the most recent formulation [22]. The probability distribution $P_\beta(q,H)$ is obtained by maximizing the entropy under two conditions,

$$\int dHP_\beta(q,H) = 1, \quad (3)$$

$$\frac{\int dH[P_\beta(q,H)]^q \mathrm{Tr}(H^\dagger H)}{\int dH[P_\beta(q,H)]^q} = \sigma_\beta^2, \quad (4)$$

where σ_β is a constant. The optimization of S_q with these constraints yields a power-law type for $P_\beta(q,H)$

$$P_\beta(q,H) = \tilde{Z}_q^{-1}[1 + (q-1)\tilde{\eta}_q\{\mathrm{Tr}(H^\dagger H) - \sigma_\beta^2\}]^{-1/(q-1)}, \quad (5)$$

where $\tilde{\eta}_q > 0$ is related to the Lagrange multiplier η associated with the constraint in Eq. (4) by

$$\tilde{\eta}_q = \eta \Big/ \int dH[P_\beta(q,H)]^q \quad (6)$$

and

$$\tilde{Z}_q = \int dH[1 + (q-1)\tilde{\eta}_q\{\mathrm{Tr}(H^\dagger H) - \sigma_\beta^2\}]^{-1/(q-1)}. \quad (7)$$

It turns out that the distribution (5) can be written hiding the presence of σ_β^2 in a more convenient form

$$P_\beta(q,H) = Z_q^{-1}[1 + (q-1)\eta_q\,\mathrm{Tr}(H^\dagger H)]^{-1/(q-1)}, \quad (8)$$

where

$$\eta_q = \frac{\eta}{\int dH[P_\beta(q,H)]^q + (1-q)\eta\sigma_\beta^2} \quad (9)$$

and

$$Z_q = \int dH[1 + (q-1)\eta_q\,\mathrm{Tr}(H^\dagger H)]^{-1/(q-1)}. \quad (10)$$

The nonextensive distribution (8) is reduced to the statistical weight of the Gaussian ensemble when $q=1$.

It is important to note that the nonextensive distribution $P_\beta(q,\eta_q,H)$ is isotropic in the Hilbert space because the dependence on the matrix elements of H enters through $\mathrm{Tr}(H^\dagger H)$. In this way, Tsallis' statistics offers a random-matrix model for mixed systems, which is invariant under change of basis unlike most of the models in the literature. However, the distribution does not factorize into a product of distributions corresponding to the individual matrix elements if $q\neq 1$. Physically, this implies that the starting hypothesis of the standard RMT that the matrix elements are independent random variables does not hold in the nonextensive context described by Eq. (2).

The formalism developed in this section was applied in Ref. [18] to ensembles of 2×2 matrices. The calculation of the spacing distribution showed different behavior depending on whether q is above or below 1. It is found that the subextensive regime of $q>1$ corresponds to the evolution of a mixed system towards a state of order described by the Poisson statistics. On the other hand, the spectrum in the superextensive regime develops towards the picked-fence type, such as the one obtained by Berry and Tabor [23] for the two-dimensional harmonic oscillator with noncommensurate frequencies.

A. Subextensive regime

In this paper, we shall consider only the subextensive regime, where $q>1$. We note that $\mathrm{Tr}(H^\dagger H)=\Sigma_{i=1}^N (H_{ii}^{(0)})^2 + 2\Sigma_{\gamma=0}^{\beta-1}\Sigma_{i>j}(H_{ij}^{(\gamma)})^2$, where all four matrices $H^{(\gamma)}$ with $\gamma=0,1,2,3$ are real and where $H^{(0)}$ is symmetric while $H^{(\gamma)}$ with $\gamma=1,2,3$ are antisymmetric. We introduce the new coordinates $\mathbf{y}=\{y_1,\ldots,y_d\}$, where $d=N+\beta N(N-1)/2$ and y_i^2 stand for the square of the diagonal elements or twice the square of the nondiagonal elements, respectively, and express the integrals in hyperspherical coordinates. The normalization condition (3) yields

$$Z_q(\eta_q) = 2^{-\beta N(N-1)/2}\Omega_d \int_0^\infty y^{d-1}dy[1+(q-1)\eta_q y^2]^{-1/(q-1)}$$
$$= 2^{-\beta N(N-1)/2}\Omega_d \frac{\Gamma\left(\frac{1}{q-1}-\frac{d}{2}\right)}{2[(q-1)\eta_q]^{d/2}\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{1}{q-1}\right)} \quad (11)$$

provided that $q<1+2/d$, otherwise the integral diverges. Here $\Omega_d=2\pi^{d/2}/\Gamma(d/2)$ is the area of a unit d-dimensional hypersphere and $\Gamma(z)$ is Euler's gamma function. Condition (4), now reads

$$\sigma_\beta^2 = \int_0^\infty y^{d+1}dy[1+(q-1)\eta_q y^2]^{-q/(q-1)} \Bigg/ \int_0^\infty y^{d-1}dy \times[1+(q-1)\eta_q y^2]^{-q/(q-1)}. \quad (12)$$

The latter yield the following relationship between η_q and σ_β^2:

$$\eta_q = \frac{d}{\sigma_\beta^2(2+d-dq)}. \quad (13)$$

For a Gaussian ensemble, $\sigma_\beta^2=2dv^2$, where v^2 is the variance of each of the nondiagonal matrix elements (or each of their components), so that $\eta_1=1/(4v^2)$. Condition (4) thus imposes the following upper limit on q:

$$q<1+\frac{2}{d}, \quad (14)$$

beyond which the nonextensive formalism is not applicable for random matrix ensembles. This condition prevents the evolution of a chaotic system towards a state of order from reaching its terminal stage of the Poisson fluctuation statistics, as we may see in [18] for the case of $N=2$, and later in this paper for the general case. The upper limit in Eq. (14) is essentially the extensive limit $(q\to 1+0)$ since RMT is meant essentially for large matrices $(d\to\infty)$. For example, the subextensive regime for a GOE of 20×20 matrices is associated with values of q in the narrow range of $1<q<1.1$. In spite of this, a minor nonextensivity produces a considerable effect on the spectral statistics of a large system as demonstrated below. This is attributed to the existence of the nontrivial "thermodynamic limit" $N(q-1)=$constant, as pointed out by Botet *et al.* [24].

B. Integral representation

In the subextensive regime, the nonextensive RMT can arise from the extensive one by allowing the variances of the matrix elements to fluctuate using a transformation suggested by Wilk and Włodarczyk [25] and Beck [26]. From Euler's representation of the gamma function [27], $\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt$, one can easily derive the following expression:

$$[1+(q-1)\eta_q \operatorname{Tr}(H^\dagger H)]^{-1/(q-1)} = \frac{1}{\Gamma\left(\frac{1}{q-1}\right)}\int_0^\infty t^{1/(q-1)-1}e^{-t[1+(q-1)\eta_q \operatorname{Tr}(H^\dagger H)]}dt, \quad (15)$$

which is possible if $q>1$. We now change the integration variable into $\eta=(q-1)\eta_q t$. The joint distribution of matrix elements of a Tsallis random-matrix ensemble can then be expressed as

$$P_\beta(q,\eta_q,H) = \int_0^\infty f_n(\eta)\frac{Z_1(\eta)}{Z_q(\eta_q)}P_\beta(\eta,H)d\eta, \quad (16)$$

where

$$P_\beta(\eta,H) = Z_1^{-1}e^{-\eta \operatorname{Tr}(H^\dagger H)}, \quad (17)$$

with

$$Z_1(\eta) = \int dHe^{-\eta \operatorname{Tr}(H^\dagger H)} = \frac{2^{-\beta N(N-1)/2}\Omega_d}{2\eta^{d/2}\Gamma(d/2)} \quad (18)$$

the distribution function for a Gaussian random-matrix ensemble with fluctuating matrix-element inverse variance $1/2\eta$, and $f_n(\eta)$ is the probability density of the χ^2-distribution (the distribution of sum of squares of n normal variables with zero mean and unit variance),

$$f_n(\eta) = \frac{1}{\Gamma(n/2)}\left(\frac{n}{2\eta_q}\right)^{n/2}\eta^{n/2-1}\exp\left(-\frac{n\eta}{2\eta_q}\right), \quad (19)$$

with order $n=2/(q-1)$ and mean value $\eta_q=nd/[2\sigma_\beta^2(n-d)]=n/[4v^2(n-d)]$. Therefore the generalized distribution function $P_\beta(q,\eta_q,H)$ of nonextensive statistics is expressed in terms of the distribution function $P_\beta(\eta,H)$ of the corresponding Gaussian random-matrix ensemble by averaging over η, provided that η has a χ^2 distribution.

As mentioned above, the nonextensive Hamiltonian matrix-element distribution in Eqs. (5) and (16) is invariant under change of basis. The mean value of each matrix element $\langle H_{ij}^{(\gamma)}\rangle=0$. On the other hand, the mean value of the square of a matrix element

$$\left\langle (H_{ij}^{(\gamma)})^2\right\rangle = \frac{1+\delta_{ij}}{4}\frac{n}{\eta_q(n-d-2)} = (1+\delta_{ij})v^2\frac{n-d}{n-d-2}, \quad (20)$$

which is equal to the corresponding quantity in the standard RMT when $n\to\infty$, as expected. The distribution does not factorize into a product of distributions of individual matrix element, or matrix-element components, as in the standard RMT. The relative dispersion of the squares of the matrix elements

$$\frac{\langle (H_{ij}^{(\gamma)})^2(H_{i'j'}^{(\gamma')})^2\rangle - \langle (H_{ij}^{(\gamma)})^2\rangle\langle (H_{i'j'}^{(\gamma')})^2\rangle}{\langle (H_{ij}^{(\gamma)})^2\rangle\langle (H_{i'j'}^{(\gamma')})^2\rangle} = \frac{2}{n-d-4} \quad (21)$$

vanishes only in the extensive limit of $n\to\infty$. We note that, for a given n and fixed v, the degree of correlation of matrix element measured by the covariance of their squares increases with increasing the dimension of the ensemble. This agrees with the result of nonextensive thermostatistics for the partition function of a system with N subsystems, which strongly suggests that the factorization approximation fails when N is large [29,30].

C. Eigenvalue distribution

We now calculate the joint probability density for the eigenvalues of the Hamiltonian H. With $H=U^{-1}XU$, where U is the global unitary group, we introduce the elements of the diagonal matrix of eigenvalues $X=\mathrm{diag}(x_1,\ldots,x_N)$ of the eigenvalues and the independent elements of U as new variables. Then the volume element (4) has the form

$$dH = |\Delta_N(X)|^{\beta} dX d\mu(U), \tag{22}$$

where $\Delta_N(X)=\Pi_{n>m}(x_n-x_m)$ is the Vandermonde determinant and $d\mu(U)$ the invariant Haar measure of the unitary group [9,12]. The probability density $P_\beta(H)$ is taken to be invariant under arbitrary rotations in the matrix space, $P_\beta(\eta,H)=P_\beta(\eta,U^{-1}HU)$. Integrating over U yields the joint probability density of eigenvalues in the form

$$P_\beta^{(q)}(\eta_q,x_1,\ldots,x_N) = \frac{\Gamma\left(\frac{n}{2}\right)}{(n/2)^{d/2}\Gamma\left(\frac{n-d}{2}\right)} \int_0^\infty f_n(\eta)\left(\frac{\eta_q}{\eta}\right)^{d/2} \times P_\beta^{(1)}(\eta,x_1,\ldots,x_N)d\eta, \tag{23}$$

where $P_\beta^{(1)}(\eta,x_1,\ldots,x_N)$ is the eigenvalue distribution of the corresponding Gaussian ensemble, which is given by

$$P_\beta^{(1)}(\eta,x_1,\ldots,x_N) = C_\beta|\Delta_N(X)|^{\beta}\exp\left[-\eta\sum_{i=1}^{N}x_i^2\right], \tag{24}$$

where C_β is a normalization constant. Similar relations can be obtained for the statistics that can be obtained from $P_\beta^{(q)}(\eta_q,x_1,\ldots,x_N)$ by integration.

The k-point correlation function [9,12] measures the probability density of finding a level near each of the positions $x_1,\ldots,x_k$, the remaining levels not being observed. It is obtained by integrating the eigenvalue joint probability density (16) over $N-k$ arguments

$$R_{\beta,k}^{(q)}(\eta_q,x_1,\ldots,x_k) = \int_{-\infty}^{\infty}dx_{k+1}\cdots\int_{-\infty}^{\infty}dx_N P_\beta^{(q)}(\eta,x_1,\ldots,x_N). \tag{25}$$

Therefore the nonextensive generalization of the k-point function of a Gaussian ensemble $R_{\beta,k}^{(1)}(\eta,x_1,\ldots,x_k)$ is given by

$$R_{\beta,k}^{(q)}(\eta_q,x_1,\ldots,x_k) = \frac{1}{\Gamma\left(\frac{m}{2}\right)}\int_0^\infty\left(\frac{n\eta}{2\eta_q}\right)^{m/2}e^{-n\eta/2\eta_q}R_{\beta,k}^{(1)} \times(\eta,x_1,\ldots,x_k)\frac{d\eta}{\eta}, \tag{26}$$

where

$$m = n - d.$$

Noting that $n/2\eta_q=2v^2m$, we can easily see that the main parameter is m, which is subject to the restriction

$$0 < m < \infty. \tag{27}$$

The lower limit follows from the normalization condition as well as the constraint of finite average matrix norm (4). The upper limit corresponds to the standard RMT.

III. LEVEL DENSITY

The main goal of RMT is to describe the fluctuations of the energy spectra. Before the study of the fluctuations can be attempted, one must make a separation between the local level fluctuation from the overall energy dependence of the level separation. The level density of the standard random matrix ensembles is not directly related to the physical level density of the investigated systems. Nevertheless, it is essential to the proper unfolding of the spectral fluctuation measures. For the N-dimensional GOE, the level density normalized to 1 is given by Wigner's semicircle law

$$\rho_1(\infty,\varepsilon) = \frac{2}{\pi}\sqrt{\eta/N}\sqrt{1-\eta\varepsilon^2/N}. \tag{28}$$

Here $\varepsilon=x/v$ is the energy expressed in units of standard deviation of the majority of matrix elements. In the following, we derive a corresponding formula for the nonextensive generalization of GOE, which we shall refer to as the Tsallis orthogonal ensemble (TsOE).

The level density is obtained by integrating the joint probability density of eigenvalues over all variables except one, so that the level density of TsOE $\rho_q(m,x)=R_{1,1}^{(q)}(\eta_q,x)$. Therefore, using Eq. (28) for $R_{1,1}^{(1)}$ into Eq. (26), we obtain

$$\rho_q(m,\varepsilon) = \frac{(2mN)^{m/2}\Gamma\left(\frac{m+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{m}{2}+2\right)}|\varepsilon|^{-m-1}{}_1F_1\left(\frac{m+1}{2},\frac{m}{2}+2,-\frac{2mN}{\varepsilon^2}\right), \tag{29}$$

where ${}_1F_1(a,b,z)$ is Kummer's confluent hypergeometric function [27]. At $\varepsilon=0$, Eq. (27) yields

$$\rho_q(m,0) = \frac{1}{\pi}\sqrt{\frac{2}{mN}}\frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m}{2}\right)}. \tag{30}$$

Using the asymptotic properties of $\Gamma(z)$ at large z [27], we can show that the height of $\rho_q(m,\varepsilon)$ at the origin is lower than the GOE level density, $\rho_1(\infty,0)=1/(\pi\sqrt{N})$, since the ratio of the gamma functions in the right-hand side of Eq. (30) is approximately equal to $\sqrt{m/2}[1-1/(4m)]$ for $m\gg1$. At small m, where the relation $\Gamma(1+z)=z\Gamma(z)$ tells us that $\Gamma(m/2)\approx 2/m$, the dependence of $\rho_q(m,0)$ on m is mainly

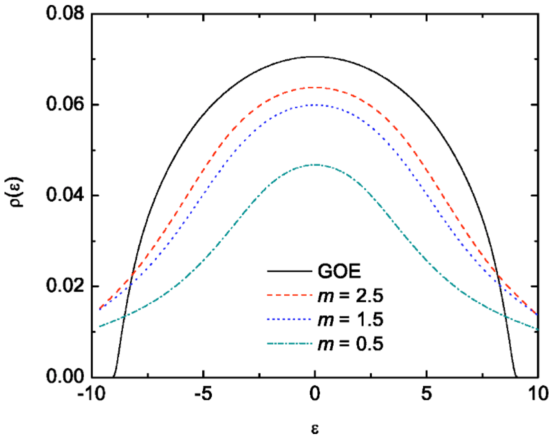

FIG. 1. The level density (normalized to 1) for TsOE plotted against energy eigenvalues expressed in units of v for different values of the parameter m.

given by the factor $\sqrt{m}$. On the other hand, the asymptotic behavior of $\rho_q(m,\varepsilon)$ at large $|\varepsilon|$ is given by

$$\rho_q(m,\varepsilon) \sim |\varepsilon|^{-m-1}. \tag{31}$$

Decreasing m from very large values lowers the magnitude of $\rho_q(m,\varepsilon)$ at the origin below the semicircle form of the Gaussian ensemble and raises its values at the periphery. This effect is clearly demonstrated in the left panel of Fig. 1. The behavior shown in this figure is similar to that of the results of calculations by Toscano *et al.* [15].

In order to perform the statistical analysis of level fluctuations of the energy levels, one must take into account that the level density and hence the level spacing are strongly dependent on the intrinsic energy. For this purpose, the investigated spectra are transformed into the so-called "unfolded" spectra [28] for which the local mean spacing is 1. On the other hand, calculations using RMT are performed for levels near the origin, where the level density is nearly equal to a constant proportional to $\sqrt{\eta}$. The energy scale is so far defined by the standard deviation of matrix elements, as, e.g., in Eq. (20). It is more suitable to express the quantities having the dimension of energy in terms of the mean level spacing rather than standard deviation of matrix elements. For this purpose we replace the ratio $n\eta/2\eta_q$ in Eq. (19) by η/η_0 and define η_0 by the requirement that the mean level spacing is 1. We then obtain for the nonextensive generalization any statistic $R_\beta^{\rm GE}$ of a Gaussian ensemble

$$R_\beta^{(m)} = \frac{1}{\Gamma\left(\frac{m}{2}\right)}\int_0^\infty \left(\frac{\eta}{\eta_0}\right)^{m/2} e^{-\eta/\eta_0} R_\beta^{\rm GE}(\eta)\frac{d\eta}{\eta}, \tag{32}$$

where η is now understood as the square of the level density.

IV. NEAREST-NEIGHBOR-SPACING DISTRIBUTION

The nearest-neighbor-spacing distribution (NNSD) is frequently used for the analysis of experimental spectra. Unfortunately, RMT does not provide a closed form expression for NNSD. A very good approximation for this distribution is given by the so-called Wigner surmise [9], which is the exact spacing distribution for Gaussian ensembles of 2×2 matrices. In this section we shall assume this approximation. We substitute the Wigner surmise for GOE and GUE into Eq. (32) and obtain expressions for NNSD of the corresponding Tallis ensembles.

A. Systems invariant under time reversal

Chaotic systems, whose Hamiltonians are invariant under time reversal, are modeled in RMT by GOE. For these ensembles, the Wigner surmise is

$$P^{\rm GOE}(\eta,s) = \eta s e^{-1/2\eta s^2}, \tag{33}$$

where η is obtained by requiring that $P^{\rm GOE}$ has a mean spacing equal to 1. Substituting Eq. (26) into Eq. (25), we obtain the following expression for the nonextensive (Tsallis) orthogonal ensemble:

$$P^{\rm TsOE}(m,s) = \frac{\frac{1}{2}m\eta_1 s}{\left(1+\frac{1}{2}\eta_1 s^2\right)^{1+m/2}}, \tag{34}$$

where

$$\eta_1 = \frac{\pi}{2}\left[\frac{\Gamma\left(\frac{1}{2}(m-1)\right)}{\Gamma\left(\frac{1}{2}m\right)}\right]^2 \tag{35}$$

is obtained by requiring that $\int_0^\infty sP^{\rm TsOE}(m,s)ds=1$.

In the limit of $m\to\infty$, $\eta_0 \approx \pi/m$ and the nonextensive NNSD approaches the extensive one, which is peaked at $s=\sqrt{2/\pi}\approx 0.80$. At the other limit where $m=1$, the mean spacing distribution diverges. If one also requires that the second moment is finite, one increases the lower limit into $m=2$, for which one obtains $\eta_0=\pi^2/2$ and the NNSD becomes

$$P^{\rm TsOE}(2,s) = \frac{\pi^2 s}{(1+\pi^2 s^2)^2}, \tag{36}$$

which vanishes at the origin, has a peak at $s=2\sqrt{2}/(\pi\sqrt{3})\approx 0.52$, and decays asymptotically as s^{-3}. We therefore see that increasing nonextensivity modifies the NNSD from a Wigner form towards a Poisson distribution e^{-s} but never reaches it. Equation (34) agrees with the corresponding result obtained for the 2×2 random matrix ensemble directly by integrating the joint eigenvalue distribution [18]. This behavior is explicitly demonstrated in that paper. We thus expect the nonextensive RMT to describe the transition out of chaos, at least in its initial stage. We note that our result for the NNSD agrees with the corresponding distributions obtained in Ref. [15] for the case of orthogonal universality.

Figure 2 compares the NNSD in Eq. (34) with the corresponding results of a numerical experiment [31]. This experiment imitates a one-parameter (denoted by δ) transition between an ensemble of diagonal matrices with independently and uniformly distributed elements and a circular orthogonal

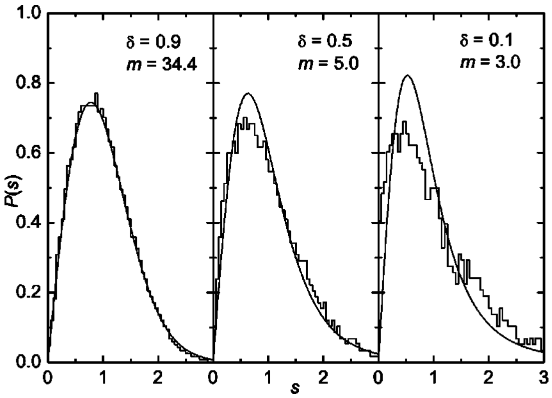

FIG. 2. NNSD for TsOE, calculated using Eq. (34) compared with the results of the numerical experiment by Życzkowski and Kuś [31].

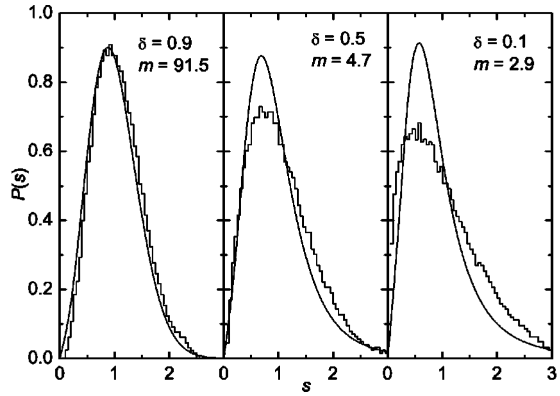

FIG. 3. NNSD for TsUE, calculated using Eq. (38) compared with the results of the numerical experiment by Życzkowski and Kuś [31].

or unitary ensemble. The figure shows that the TsOE distributions are generally in agreement with the numerical-experimental distribution although the quality of agreement gradually deteriorates, as expected, as the departure from chaos increases.

B. Systems without time reversal symmetry

RMT models systems whose Hamiltonians violate time reversibility by GUE. The corresponding Wigner surmise is

$$P^{\mathrm{GUE}}(\eta,s)=\sqrt{\frac{2}{\pi}}\eta^{3/2}s^2e^{-1/2\eta s^2}, \tag{37}$$

where η is obtained by requiring that P^{GUE} has a mean spacing equal to 1. Substituting Eq. (37) into Eq. (32), we obtain the following expression for the nonextensive (Tsallis) unitary ensemble

$$P^{\mathrm{TsUE}}(m,s)=\sqrt{\frac{2}{\pi}}\frac{\Gamma\left(\frac{1}{2}(3+m)\right)}{\Gamma\left(\frac{1}{2}m\right)}\frac{\eta_2^{3/2}s^2}{\left(1+\frac{1}{2}\eta_2s^2\right)^{(3+m)/2}}, \tag{38}$$

where

$$\eta_2=\frac{8}{\pi}\left[\frac{\Gamma\left(\frac{1}{2}(m-1)\right)}{\Gamma\left(\frac{1}{2}m\right)}\right]^2 \tag{39}$$

is obtained by requiring that $\int_0^\infty sP^{\mathrm{TsUE}}(m,s)ds=1$. The evolution of the distribution $P^{\mathrm{TsUE}}(s)$ as m decreases from ∞, where it is given by the Wigner surmise, at the limiting value of $m=2$ is demonstrated in Ref. [18] for the two-dimensional case. Figure 3 compares the NNSD for the TsUE with the corresponding distributions in the numerical experiment in Ref. [31]. We again see that the proposed nonextensive RMT provides a satisfactory description for the stochastic transition in terms of a single parameter, particularly in its early stage [32].

V. CONCLUSION

In the present work we have obtained a nonextensive generalization of the matrix-element theory by extremizing Tsallis's entropy, indexed by q, subjected to two constraints: normalization and finite average norm of the matrices. We consider the subextensive regime of $q>1$, where the transition from chaos to order described by the Poisson statistics is expected. The constraint of finite matrix-norm forces an upper limit of the entropic index, limiting the attainable range to $1<q<1+2/d$, where d is the dimension of the matrix-element space. This is essentially the extensive limit since RMT normally involves large matrices. Nonetheless, the obtained fluctuation statistics depend mainly on the parameter $m=-d+2/(q-1)$, so that a minor deviation from extensivity leads to an observable effect. Because of this limitation, we expect the nonextensive formalism to provide a description for the early stages of transition from chaos towards regularity. We obtain distribution functions for the three symmetry universality classes, for which the probability of larger matrix elements decay algebraically instead of exponentially. By means of an integral transform, which is based on an integral representation of the gamma function, we express the characteristics of the nonextensive theory to those of the standard RMT. We calculate the level density and the NNSD for systems with mixed regular chaotic dynamics. We have also derived a generalization for the Wigner surmise that can be compared to numerical experiments with mixed systems.

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